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HOMOTOPY GROUPS OF THE SPACES OF SELF-MAPS OF LIE GROUPS

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ABSTRACT. We compute the homotopy groups of the spaces of self maps of Lie groups of rank 2, SU(3), Sp(2), and G_2. We use the cell structures of these Lie groups and the standard methods of homotopy theory.

1. Introduction

For pointed spaces X and Y, we let $\mathrm{map}_*(X,Y)$ denote the space of pointed maps from X to Y. We take the trivial map $*$ as a base point of $\mathrm{map}_*(X,Y)$. The homotopy groups of function spaces have long been studied in homotopy theory. Indeed, if $X = \mathrm{S}^n$, then $\mathrm{map}_*(\mathrm{S}^n, Y)$ coincides with the iterated loop space $\Omega^n Y$. Hence the homotopy groups $\pi_n \mathrm{map}_*(\mathrm{S}^n, Y)$ are known by the homotopy groups of Y. However, even if the number of the cells of X is small, the determination of the group structure of $\pi_n \mathrm{map}_*(X,Y)$ is not easy in general.

In this paper we study the homotopy groups of the self maps $\mathrm{map}_*(X,X)$ in the case where X is a compact Lie group of rank 2. Precisely, we consider SU(3), Sp(2), and G_2. The homotopy-theoretic structures of these spaces are well known. In particular, their homotopy groups are computed in Mimura-Toda [**MT**], and Mimura [**M**]. Our results entirely depend on their work.

The homotopy groups of $\mathrm{map}_*(X,X)$ are closely related to the homotopy groups of other interesting spaces. For instance, we have

(i) We can apply our results to the homotopy groups of the spaces of self-homotopy equivalences. When X is a topological group, all connected components of $\mathrm{map}_*(X,X)$ have the same homotopy type. Hence we have an isomorphism:

$$\pi_n(\mathrm{aut}_*(X), 1_X) \cong \pi_n \mathrm{map}_*(X,X)$$

where $\mathrm{aut}_*(X)$ is the space of the based maps of X which are homotopy equivalences. In [**D**], Didierjean studied the homotopy groups of $\pi_n(\mathrm{aut}_*(X))$ for rank 2 Lie groups by using other methods. Our results in this paper extend some of the results in [**D**].

(ii) Our results in this paper can be used to know the homotopy types of the gauge groups $\mathcal{G}(P)$. Generally, for a principal G-bundle $P \to X$,

$$\mathrm{map}_P(X, BG) \simeq B\mathcal{G}(P)$$

by Atiyah-Bott [**AB**], where $\mathrm{map}_P(X, BG)$ is a subspace of $f \in \mathrm{map}(X, BG)$ such that f is homotopic to the classifying map of P. There exists a fibration as follows.

$$G \xrightarrow{\alpha} \mathrm{map}_{*,P}(X, BG) \to B\mathcal{G}(P) \to BG,$$

where $\mathrm{map}_{*,P}(X, BG) = \mathrm{map}_*(X, BG) \cap \mathrm{map}_P(X, BG)$. In particular, when $X = \mathrm{S}^n$, the adjoint of the map α is an element of $\pi_{n-1}\mathrm{map}_*(G,G)$.

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Finally, we make mention of the homotopy group $\pi_0\mathrm{map}_*(X,X)$. This set is considered as the homotopy classes $[X,X]$, and is a group when X is a topological group. In the case that X is a connected Lie group of rank 2, $\pi_0\mathrm{map}_*(X,X)$ are studied in [**AOS**, **KO**, **MO**, **O1**, **O2**, **O3**].

Now we state our main results in this paper.

Theorem 1.

n	$\pi_n\mathrm{map}_*(\mathrm{SU}(3),\mathrm{SU}(3))$	$\pi_n\mathrm{map}_*(\mathrm{Sp}(2),\mathrm{Sp}(2))$
1	$\mathbb{Z}_3^2$	$\mathbb{Z}_2^2$
2	$\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_5$	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_4\oplus\mathbb{Z}_8\oplus\mathbb{Z}_3^2$	$\mathbb{Z}_2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8\oplus\mathbb{Z}_5$
4	$\mathbb{Z}_4\oplus\mathbb{Z}_3^2\oplus\mathbb{Z}_5$	$\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{16}\oplus\mathbb{Z}_3\oplus\mathbb{Z}_5\oplus\mathbb{Z}_7$
5	$\mathbb{Z}_2\oplus A\oplus\mathbb{Z}_3^3\oplus\mathbb{Z}_5$	$\mathbb{Z}_2^3$
6	$\mathbb{Z}_2\oplus\mathbb{Z}_4^2\oplus\mathbb{Z}_3^2\oplus\mathbb{Z}_7$	$\mathbb{Z}_2^4$
7	$\mathbb{Z}_4\oplus\mathbb{Z}_8\oplus\mathbb{Z}_3^2\oplus\mathbb{Z}_9\oplus\mathbb{Z}_5^2$	$\mathbb{Z}_8\oplus\mathbb{Z}_{32}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_9\oplus\mathbb{Z}_5^3\oplus\mathbb{Z}_7$
8	$\mathbb{Z}_2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8\oplus\mathbb{Z}_3^2\oplus\mathbb{Z}_9\oplus\mathbb{Z}_7$	$\mathbb{Z}_2^3\oplus\mathbb{Z}_8\oplus\mathbb{Z}_9\oplus\mathbb{Z}_5\oplus\mathbb{Z}_7$

Here $\mathbb{Z}_n^r$ denotes the direct sum of r copies of $\mathbb{Z}_n$, and A is $\mathbb{Z}_2\oplus\mathbb{Z}_4$ or $\mathbb{Z}_8$. Hamanaka-Kono [**HK**] proves $A=\mathbb{Z}_8$.

For the exceptional Lie group G_2 we obtain the following.

Theorem 2. $\pi_1\mathrm{map}_*(G_2,G_2)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$.

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2. Preliminaries

As defined in the introduction, $\mathrm{map}_*(X,Y)$ denote the function space of pointed maps from X to Y. We consider $\mathrm{map}_*(X,Y)$ as a topological space having the compact open topology. We denote by $\pi_n\mathrm{map}_*(X,Y)$ the homotopy group of the component of the trivial map. Namely,

$$\pi_n\mathrm{map}_*(X,Y)=\pi_n(\mathrm{map}_*(X,Y),*).$$

In this paper we shall identify $\pi_n\mathrm{map}_*(X,Y)$ with $[\Sigma^nX,Y]$ by the adjoint isomorphism, where $\Sigma^nX=\mathrm{S}^n\wedge X$.

Recall that if the following diagram is commutative up to homotopy, then we call $\overline{h}$ an *extension* of h and $\widetilde{f}$ a *coextension* of f.

$$\begin{tikzcd}
W \arrow[dr, "f"] & & & \Sigma W \arrow[d, "\widetilde{f}"'] \arrow[dr, "-\Sigma f"] & \\
& X \arrow[r, "g"] & Y \arrow[r, "i"] \arrow[dr, "h"'] & C_g \arrow[r, "q"'] \arrow[d, "\overline{h}"] & \Sigma X \\
& & & Z &
\end{tikzcd}$$

Here $C_g = Y \cup_g CX$ is the reduced mapping cone of g, i is the inclusion, and q is the quotient map.

We follow Toda's notation [**T2**] for elements of homotopy groups of spheres.

As is well-known, we have

$$\mathrm{SU}(3) = \mathrm{S}^3 \cup_{\eta_3} e^5 \cup_\phi e^8, \quad \pi_4(\mathrm{S}^3) = \mathbb{Z}_2\{\eta_3\};$$
$$\mathrm{Sp}(2) = \mathrm{S}^3 \cup_\omega e^7 \cup e^{10}, \quad \pi_6(\mathrm{S}^3) = \mathbb{Z}_{12}\{\omega\}, \quad \omega = \nu' + \alpha_1(3).$$

Let

$$\mathrm{S}^3 \xrightarrow{i'} C_{\eta_3} \xrightarrow{j} \mathrm{SU}(3); \quad \mathrm{S}^3 \xrightarrow{i'} C_\omega \xrightarrow{j} \mathrm{Sp}(2)$$

be the inclusion maps. Write $i = j \circ i'$. Let

$$q_3 : C_{\eta_3} \to \mathrm{S}^5, \quad q : \mathrm{SU}(3) \to \mathrm{S}^8; \quad q_3 : C_\omega \to \mathrm{S}^7, \quad q : \mathrm{Sp}(2) \to \mathrm{S}^{10}$$

be the quotient maps. Let

$$\mathrm{S}^3 \xrightarrow{i} \mathrm{SU}(3) \xrightarrow{p} \mathrm{S}^5 \ ; \qquad \mathrm{S}^3 \xrightarrow{i} \mathrm{Sp}(2) \xrightarrow{p} \mathrm{S}^7$$

be the canonical fibrations. As is well-known, $p \circ j = q_3$.

Notation 2.1. *Given* $x \in [\Sigma^m C_{\eta_3}, \mathrm{SU}(3)]$ *(resp.* $x \in [\Sigma^m C_\omega, \mathrm{Sp}(2)]$*), an extension of* x *to* $\Sigma^m \mathrm{SU}(3)$ *(resp.* $\Sigma^m \mathrm{Sp}(2)$*) is denoted by* $\overline{x} \in [\Sigma^m \mathrm{SU}(3), \mathrm{SU}(3)]$ *(resp.* $\overline{x} \in [\Sigma^m \mathrm{Sp}(2), \mathrm{Sp}(2)]$*), that is,* $x = \left(\Sigma^m j\right)^* \overline{x}$*. Given* $z \in [\Sigma^m \mathrm{S}^3, \mathrm{SU}(3)]$ *(resp.* $z \in [\Sigma^m \mathrm{S}^3, \mathrm{Sp}(2)]$*), we denote by* $\overline{\overline{z}}$ *an element of* $[\Sigma^m \mathrm{SU}(3), \mathrm{SU}(3)]$ *(resp.* $[\Sigma^m \mathrm{Sp}(2), \mathrm{Sp}(2)]$*) such that* $z = \left(\Sigma^m i\right)^* (\overline{\overline{z}})$.

$$\begin{tikzcd}
\Sigma^m C_{\eta_3} \arrow[r, "\Sigma^m j"] \arrow[dr, dotted, "x"] & \Sigma^m \mathrm{SU}(3) \arrow[d, dotted, shift right, "\overline{x}"'] \arrow[d, shift left, bend left, "\overline{\overline{z}}"] \\
\Sigma^m \mathrm{S}^3 \arrow[u, "\Sigma^m i'"] \arrow[r, "z"'] & \mathrm{SU}(3)
\end{tikzcd} ; \qquad
\begin{tikzcd}
\Sigma^m C_{\omega} \arrow[r, "\Sigma^m j"] \arrow[dr, dotted, "x"] & \Sigma^m \mathrm{Sp}(2) \arrow[d, dotted, shift right, "\overline{x}"'] \arrow[d, shift left, bend left, "\overline{\overline{z}}"] \\
\Sigma^m \mathrm{S}^3 \arrow[u, "\Sigma^m i'"] \arrow[r, "z"'] & \mathrm{Sp}(2)
\end{tikzcd}$$

For any abelian group Γ and a set of prime numbers P, let $\Gamma_{(P)}$ be the localization of Γ at P. Given maps $f : X \to Y$ and $g : Y \to Z$, we usually denote their composition by $g \circ f$, but sometimes we denote it simply by gf.

3. $\pi_n \mathrm{map}_*(\mathrm{SU}(3), \mathrm{SU}(3))$

The odd primary components of $[\Sigma^n \mathrm{SU}(3), \mathrm{SU}(3)]$ are easily obtained from the results in [**T2**], since if p is an odd prime, then $\mathrm{SU}(3)_{(p)} \simeq \mathrm{S}^3_{(p)} \times \mathrm{S}^5_{(p)}$ (homotopy equivalent). Thus

$$[\Sigma^n \mathrm{SU}(3), \mathrm{SU}(3)]_{(p)} \cong \pi_{n+3}(\mathrm{S}^3 \times \mathrm{S}^5)_{(p)} \oplus \pi_{n+5}(\mathrm{S}^3 \times \mathrm{S}^5)_{(p)} \oplus \pi_{n+8}(\mathrm{S}^3 \times \mathrm{S}^5)_{(p)}. \tag{3.1}$$

Hence in the rest of this section we calculate $[\Sigma^n \mathrm{SU}(3), \mathrm{SU}(3)]_{(2)}$ for $n \geq 1$. We use

n	$\pi_n \mathrm{SU}(3)$	gen. of 2-comp.	n	$\pi_n \mathrm{SU}(3)$	gen. of 2-comp.
$1,2,4,7$	0		12	$\mathbb{Z}_4 \oplus \mathbb{Z}_{15}$	$[\sigma''']$ $(2[\sigma'''] = i_*\mu_3)$
3	$\mathbb{Z}$	$i_*\iota_3$	13	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$i_*\varepsilon'$
5	$\mathbb{Z}$	$[2\iota_5]$	14	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{21}$	$[\nu_5^2]\nu_{11},\ i_*\mu'$
6	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$i_*\nu'$	15	$\mathbb{Z}_4 \oplus \mathbb{Z}_9$	$[2\iota_5]\nu_5\sigma_8$
8	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$[2\iota_5]\nu_5$	16	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{63} \oplus \mathbb{Z}_3$	$[2\iota_5]\zeta_5,\ [\nu_5\overline{\nu}_8]$
9	$\mathbb{Z}_3$		17	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{15}$	$[\nu_5]\nu_{11}^2,\ [\nu_5\eta_8\varepsilon_9]$
10	$\mathbb{Z}_2 \oplus \mathbb{Z}_{15}$	$[\nu_5\eta_8^2]$	18	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_3$	$i_*\overline{\varepsilon}_3,\ [\nu_5\eta_8\mu_9]$
11	$\mathbb{Z}_4$	$[\nu_5^2]$ $(2[\nu_5^2] = i_*\varepsilon_3)$	19	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^2$	$[\sigma''']\sigma_{12},\ [\nu_5\overline{\nu}_8]\nu_{16}$

Table 1 : $\pi_n(\mathrm{SU}(3))$

This is contained in [**MT**] with the following notation: $[x] \in \pi_n(\mathrm{SU}(3))$ denotes an element such that $p_*[x] = x$.

Fist we prove $[\Sigma\,\mathrm{SU}(3), \mathrm{SU}(3)]_{(2)} = 0$. By Table 1, we have the following exact sequence.

$$0 \xrightarrow{(\Sigma q)^*} [\Sigma\,\mathrm{SU}(3), \mathrm{SU}(3)]_{(2)} \xrightarrow{(\Sigma j)^*} [\mathrm{S}^4 \cup_{\eta_4} e^6, \mathrm{SU}(3)]_{(2)}$$

It suffices for our purpose to prove

$$[\mathrm{S}^4 \cup_{\eta_4} e^6, \mathrm{SU}(3)]_{(2)} = 0. \tag{3.2}$$

By Table 1 we have the following exact sequence.

$$\mathbb{Z}_{(2)}\{[2\iota_5]\} \xrightarrow{\eta_5^*} \mathbb{Z}_2\{i_*\nu'\} \xrightarrow{(\Sigma q_3)^*} [\mathrm{S}^4 \cup_{\eta_4} e^6, \mathrm{SU}(3)]_{(2)} \xrightarrow{(\Sigma i')^*} 0. \tag{3.3}$$

We use the following theorem [**MT**, Theorem 2.1].

Theorem 3.1 ([**MT**]). *Let $F \xrightarrow{i} X \xrightarrow{p} B$ be a fibration, and $\partial : \pi_n(B) \to \pi_{n-1}(F)$ the boundary operator. Assume that $\alpha \in \pi_{m+1}(B)$, $\beta \in \pi_l(\mathrm{S}^m)$ and $\gamma \in \pi_k(\mathrm{S}^l)$ satisfying $\partial\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. For an arbitrary element $\delta \in \{\partial\alpha, \beta, \gamma\} \subset \pi_{k+1}(F)$, there exists an element $\epsilon \in \pi_{l+1}(X)$ such that $p_*\epsilon = \alpha \circ \Sigma\beta$ and $i_*\delta = \epsilon \circ \Sigma\gamma$.*

We apply this theorem to the fibration $\mathrm{S}^3 \xrightarrow{i} \mathrm{SU}(3) \xrightarrow{p} \mathrm{S}^5$ by taking

$$\alpha = \iota_5, \quad \beta = 2\iota_4, \quad \gamma = \eta_4, \quad k = 5, \quad l = m = 4.$$

Indeed this case can be applied, since $\beta \circ \gamma = 0$ and $\partial\alpha = \eta_3$ so that $\partial\alpha \circ \beta = 0$. It follows that for any $\delta \in \{\partial\alpha, \beta, \gamma\}$ there exists $\epsilon \in \pi_5(\mathrm{SU}(3))$ such that

$$p_*\epsilon = \alpha \circ \Sigma\beta = 2\iota_5, \quad i_*\delta = \epsilon \circ \Sigma\gamma.$$

In particular we have $\epsilon = [2\iota_5]$. Since $\{\eta_3, 2\iota_4, \eta_4\} = \{\nu', -\nu'\}$ by [**T2**, (5.4)], we then have

$$i_*\nu' = [2\iota_5] \circ \eta_5 = \eta_5^*[2\iota_5]. \tag{3.4}$$

Hence by (3.3) we have (3.2) as desired.

In order to calculate $[\Sigma^n \mathrm{SU}(3), \mathrm{SU}(3)]_{(2)}$ for $n \geq 2$, we recall a result of Browder-Spanier [**BS**] that the attaching map of the top cell of an H-space is stably trivial. Hence

$$\Sigma^3 \mathrm{SU}(3) \simeq \mathrm{S}^6 \cup_{\eta_6} e^8 \vee \mathrm{S}^{11}\,. \tag{3.5}$$

More precisely, we can prove

$$\Sigma\phi = \Sigma i' \circ \nu_4 \circ \eta_7.$$

We do not use this equality in this paper. So we omit its proof. We have

Lemma 3.2. $[\Sigma^n \mathrm{SU}(3), \mathrm{SU}(3)] \cong \pi_{8+n}(\mathrm{SU}(3)) \oplus [C_{\eta_{3+n}}, \mathrm{SU}(3)]$ *for* $n \geq 2$.

Proof. If $n \geq 3$, then the result follows from (3.5). For $n = 2$, we have

$$[\Sigma^2 \mathrm{SU}(3), \mathrm{SU}(3)] \cong [\Sigma^3 \mathrm{SU}(3), B\,\mathrm{SU}(3)]$$

and the lemma follows also from (3.5). □

Hence it suffices for our purpose to determine $[C_{\eta_{3+n}}, \mathrm{SU}(3)]_{(2)}$ for $n \geq 2$. The generators of the 2-components of $[\Sigma^n \mathrm{SU}(3), \mathrm{SU}(3)]$ are as follows.

n	2-*components*	*generators*
1	0	
2	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\overline{\overline{2[2\iota_5]}}$, $(\Sigma^2 q)^*[\nu_5\eta_8^2]$
3	$\mathbb{Z}_4 \oplus \mathbb{Z}_8$	$(\Sigma^3 q)^*[\nu_5^2]$, $\overline{\overline{i_*\nu'}}$
4	$\mathbb{Z}_4$	$(\Sigma^4 q)^*[\sigma''']$
5	$\mathbb{Z}_2 \oplus \mathbb{Z}_8$	$(\Sigma^5 q)^* i_*\varepsilon'$, $\overline{\overline{[2\iota_5] \circ \nu_5}}$
6	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$	$(\Sigma^6 q)^* i_*\mu'$, $(\Sigma^6 q)^*([\nu_5^2] \circ \nu_{11})$, $\overline{\Sigma^6 q_3{}^*[\nu_5^2]}$
7	$\mathbb{Z}_4 \oplus \mathbb{Z}_8$	$(\Sigma^7 q)^*([2\iota_5] \circ \nu_5\sigma_8)$, $\overline{\overline{[\nu_5\eta_8^2]}}$,
8	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$	$(\Sigma^8 q)^*[\nu_5\bar{\nu}_8]$, $(\Sigma^8 q)^*([2\iota_5] \circ \zeta_5)$, $\overline{\overline{[\nu_5^2]}}$

Table 2 : 2-components of $[\Sigma^n \mathrm{SU}(3), \mathrm{SU}(3)]$

3.1. $[C_{\eta_5}, \mathrm{SU}(3)]$**.** By Table 1, we have the following exact sequence.

$$0 \longrightarrow [\mathrm{S}^5 \cup_{\eta_5} e^7, \mathrm{SU}(3)] \longrightarrow \mathbb{Z}\big\{[2\iota_5]\big\} \xrightarrow{\eta_5^*} \mathbb{Z}_2\{i_*\nu'\} \oplus \mathbb{Z}_3$$

Hence by (3.4) we have $[C_{\eta_5}, \mathrm{SU}(3)] = \mathbb{Z}\{\overline{2[2\iota_5]}\}$. Thus we obtain

$$[\Sigma^2 \mathrm{SU}(3), \mathrm{SU}(3)] = \mathbb{Z}\Big\{\overline{\overline{2[2\iota_5]}}\Big\} \oplus \mathbb{Z}_2\big\{\big(\Sigma^2 q\big)^*[\nu_5\eta_8^2]\big\} \oplus \mathbb{Z}_{15}.$$

3.2. $[C_{\eta_6}, \mathrm{SU}(3)]_{(2)}$. By [**T2**] and Table 1, we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
\mathbb{Z}_2\{\nu'\eta_6\} \arrow[r, "\eta_7^*", "\cong"'] \arrow[d] & \mathbb{Z}_2\{\nu'\eta_6^2\} \arrow[r] \arrow[d] & {[C_{\eta_6}, \mathrm{S}^3]_{(2)}} \arrow[r] \arrow[d, "i_*"] & \mathbb{Z}_4\{\nu'\} \arrow[r, "\eta_6^*"] \arrow[d, "i_*"] & \mathbb{Z}_2\{\nu'\eta_6\} \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}_4\big\{[2\iota_5]\nu_5\big\} \arrow[r, "(\Sigma^3 q_3)^*"] \arrow[d, "p_*"] & {[C_{\eta_6}, \mathrm{SU}(3)]_{(2)}} \arrow[r, "(\Sigma^3 i')^*"] \arrow[d, "p_*"] & \mathbb{Z}_2\{i_*\nu'\} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}_2\{\eta_5^2\} \arrow[r, "\eta_7^*"] & \mathbb{Z}_8\{\nu_5\} \arrow[r, "(\Sigma^3 q_3)^*"] & {[C_{\eta_6}, \mathrm{S}^5]_{(2)}} \arrow[r] & \mathbb{Z}_2\{\eta_5\} \arrow[r, "\eta_6^*", "\cong"'] & \mathbb{Z}_2\{\eta_5^2\}
\end{tikzcd}$$

By the first and third rows, we have the following results ([**KMNST**, Propositions 3.3 and 3.1]):

$$[C_{\eta_6}, \mathrm{S}^3]_{(2)} = \mathbb{Z}_2\big\{\overline{2\nu'}\big\}, \quad [C_{\eta_6}, \mathrm{S}^5]_{(2)} = \mathbb{Z}_4\big\{(\Sigma^3 q_3)^*\nu_5\big\}. \tag{3.6}$$

By the second row, the order of $[C_{\eta_6}, \mathrm{SU}(3)]_{(2)}$ is 8. Hence the middle column is short exact by (3.6). Since

$$p_*(\Sigma^3 q_3)^*([2\iota_5] \circ \nu_5) = (\Sigma^3 q_3)^* p_*([2\iota_5] \circ \nu_5) = 2(\Sigma^3 q_3)^*\nu_5,$$

we have $[C_{\eta_6}, \mathrm{SU}(3)]_{(2)} \ncong \mathbb{Z}_4 \oplus \mathbb{Z}_2$. Hence $[C_{\eta_6}, \mathrm{SU}(3)]_{(2)} = \mathbb{Z}_8\big\{\overline{i_*\nu'}\big\}$.

3.3. $[C_{\eta_7}, \mathrm{SU}(3)]_{(2)}$. By Table 1, we easily see that $[C_{\eta_7}, \mathrm{SU}(3)]_{(2)} = 0$.

3.4. $[C_{\eta_8}, \mathrm{SU}(3)]_{(2)}$. By Table 1, we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z}_2\big\{[\nu_5\eta_8^2]\big\} \xrightarrow{(\Sigma^5 q_3)^*} [C_{\eta_8}, \mathrm{SU}(3)]_{(2)} \xrightarrow{(\Sigma^5 i')^*} \mathbb{Z}_4\big\{[2\iota_5] \circ \nu_5\big\} \longrightarrow 0$$

This does not split as shown by Hamanaka-Kono [**HK**]. Hence

$$[C_{\eta_8}, \mathrm{SU}(3)]_{(2)} = \mathbb{Z}_8\big\{\overline{[2\iota_5] \circ \nu_5}\big\}.$$

3.5. $[C_{\eta_9}, \mathrm{SU}(3)]_{(2)}$. By Table 1, we have the following exact sequence:

$$\mathbb{Z}_2\big\{[\nu_5\eta_8^2]\big\} \xrightarrow{\eta_{10}^*} \mathbb{Z}_4\big\{[\nu_5^2]\big\} \xrightarrow{(\Sigma^6 q_3)^*} [C_{\eta_9}, \mathrm{SU}(3)]_{(2)} \longrightarrow 0$$

Thus $\eta_{10}^*[\nu_5\eta_8^2]$ is 0 or $2[\nu_5^2]$. To induce a contradiction, assume $\eta_{10}^*[\nu_5\eta_8^2] = 2[\nu_5^2]$. Then $2\big([\nu_5^2] \circ \nu_{11}\big) = \big(2[\nu_5^2]\big) \circ \nu_{11} = [\nu_5\eta_8^2] \circ \eta_{10} \circ \nu_{11} = 0$ since $\eta_{10} \circ \nu_{11} = 0$ by [**T2**]. This contradicts the fact that the order of $[\nu_5^2] \circ \nu_{11}$ is 4. Hence

$$[\nu_5\eta_8^2] \circ \eta_{10} = 0 \tag{3.7}$$

so that

$$[C_{\eta_9}, \mathrm{SU}(3)]_{(2)} = \mathbb{Z}_4\big\{(\Sigma^6 q_3)^*[\nu_5^2]\big\}.$$

3.6. $[C_{\eta_{10}}, \mathrm{SU}(3)]_{(2)}$. The purpose of this subsection is to prove

$$[C_{\eta_{10}}, \mathrm{SU}(3)]_{(2)} = \mathbb{Z}_8\big\{\overline{[\nu_5\eta_8^2]}\big\}. \tag{3.8}$$

By [**T2**], Table 1 and (3.7), we have the following commutative diagram with exact rows and columns:
(3.9)

$$\begin{tikzcd}
\mathbb{Z}_2\{\varepsilon_3\} \arrow[r, "\eta_{11}^*"] \arrow[d] & \mathbb{Z}_2^2\{\varepsilon_3\eta_{11}, \mu_3\} \arrow[r, "(\Sigma^7 q_3)^*"] \arrow[d, "i_*"] & {[C_{\eta_{10}}, \mathrm{S}^3]_{(2)}} \arrow[r] \arrow[d, "i_*"] & 0 & \\
\mathbb{Z}_4\big\{[\nu_5^2]\big\} \arrow[r, "\eta_{11}^*"] \arrow[d] & \mathbb{Z}_4\big\{[\sigma''']\big\} \arrow[r, "(\Sigma^7 q_3)^*"] \arrow[d] & {[C_{\eta_{10}}, \mathrm{SU}(3)]_{(2)}} \arrow[r, "(\Sigma^7 i')^*"] \arrow[d, "p_*"] & \mathbb{Z}_2\big\{[\nu_5\eta_8^2]\big\} \arrow[r, "\eta_{10}^*"] \arrow[d, "p_*", "\cong"'] & 0 \\
\mathbb{Z}_2\{\nu_5^2\} \arrow[r, "\eta_{11}^*=0"] & \mathbb{Z}_2\{\sigma'''\} \arrow[r, "(\Sigma^7 q_3)^*"] & {[C_{\eta_{10}}, \mathrm{S}^5]_{(2)}} \arrow[r, "(\Sigma^7 i')^*"] & \mathbb{Z}_2\{\nu_5\eta_8^2\} \arrow[r, "\eta_{10}^*"] & 0
\end{tikzcd}$$

By the first row, we have the following result ([**KMNST**, Proposition 3.7]):

$$[C_{\eta_{10}}, \mathrm{S}^3]_{(2)} = \mathbb{Z}_2\big\{(\Sigma^7 q_3)^*\mu_3\big\}. \tag{3.10}$$

We need

Proposition 3.3. (1) $[\nu_5^2] \circ \eta_{11} = 0$.
(2) ([**KMNST**, Proposition 3.5]) $[C_{\eta_{10}}, \mathrm{S}^5]_{(2)} = \mathbb{Z}_4\big\{\overline{\nu_5\eta_8^2}\big\}$.

Before proving this proposition, we prove (3.8) by using it. By Proposition 3.3, we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & \mathbb{Z}_2\{\mu_3\} \arrow[r, "(\Sigma^7 q_3)^*", "\cong"'] \arrow[d, "i_*"] & \mathbb{Z}_2\big\{(\Sigma^7 q_3)^*\mu_3\big\} \arrow[d, "i_*"] & & \\
0 \arrow[r] & \mathbb{Z}_4\big\{[\sigma''']\big\} \arrow[r, "(\Sigma^7 q_3)^*"] \arrow[d, "p_*"] & {[C_{\eta_{10}}, \mathrm{SU}(3)]_{(2)}} \arrow[r, "(\Sigma^7 i')^*"] \arrow[d, "p_*"] & \mathbb{Z}_2\big\{[\nu_5\eta_8^2]\big\} \arrow[r] \arrow[d, "p_*", "\cong"'] & 0 \\
0 \arrow[r] & \mathbb{Z}_2\{\sigma'''\} \arrow[r, "(\Sigma^7 q_3)^*"] & \mathbb{Z}_4\Big\{\overline{\nu_5\eta_8^2}\Big\} \arrow[r, "(\Sigma^7 i')^*"] & \mathbb{Z}_2\{\nu_5\eta_8^2\} \arrow[r] & 0
\end{tikzcd}$$

Hence $[C_{\eta_{10}}, \mathrm{SU}(3)]_{(2)}$ is isomorphic to $\mathbb{Z}_8$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. To induce a contradiction, assume it is $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. Then

$$[C_{\eta_{10}}, \mathrm{SU}(3)]_{(2)} = \mathbb{Z}_4\Big\{\overline{[\nu_5\eta_8^2]}\Big\} \oplus \mathbb{Z}_2\Big\{\overline{[\nu_5\eta_8^2]} - (\Sigma^7 q_3)^*[\sigma''']\Big\}$$

since $p_*\overline{[\nu_5\eta_8^2]}$ generates $[C_{\eta_{10}}, \mathrm{S}^5]_{(2)}$. We have $i_*(\Sigma^7 q_3)^*\mu_3 = 2(\Sigma^7 q_3)^*[\sigma'''] = 2\,\overline{[\nu_5\eta_8^2]}$. Hence the cokernel of the second i_* which is isomorphic to $[C_{\eta_{10}}, \mathrm{S}^5]_{(2)}$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This contradicts Proposition 3.3 (2). Therefore we obtain (3.8).

Proof of Proposition 3.3. The assertion (2) is proved in [**KMNST**, Proposition 3.5 (4)]. We prove (1) as follows. Since η_{11} is of order 2, $[\nu_5^2] \circ \eta_{11}$ is 0 or $2[\sigma''']$. To induce a contradiction, assume $[\nu_5^2] \circ \eta_{11} = 2[\sigma''']$. Then, by [**T2**, Lemma 6.4] and Table 1, we have

$$[\nu_5^2] \circ \sigma_{11} \circ \eta_{18} = [\nu_5^2] \circ \eta_{11} \circ \sigma_{12} = 2([\sigma'''] \circ \sigma_{12}) \neq 0. \tag{3.11}$$

By Table 1, we can write $[\nu_5^2]\circ\sigma_{11} = a\cdot i_*\overline{\varepsilon}_3 + b\cdot[\nu_5\eta_8\mu_9]$ $(a,b\in\mathbb{Z})$. Then

$$\nu_5^2\sigma_{11} = p_*([\nu_5^2]\circ\sigma_{11}) = b\cdot\nu_5\eta_8\mu_9.$$

By [**T2**, (7.19)], $\sigma'\nu_{14} = x\cdot\nu_7\sigma_{10}$ with x odd. Hence

$$\nu_5\circ\Sigma\sigma'\circ\nu_{15} = \nu_5\circ x\cdot\nu_8\circ\sigma_{11} = \nu_5^2\sigma_{11}.$$

On the other hand, $\nu_5\circ\Sigma\sigma' = 2(\nu_5\sigma_8)$ by [**T2**, (7.16)]. Hence $\nu_5\circ\Sigma\sigma'\circ\nu_{15} = 0$, since $2\pi_{18}(\mathrm{S}^5)_{(2)} = 0$ by [**T2**]. Thus $\nu_5^2\sigma_{11} = 0$ so that b is even and $[\nu_5^2]\circ\sigma_{11} = a\cdot i_*\overline{\varepsilon}_3$. We then have

$$[\nu_5^2]\circ\sigma_{11}\circ\eta_{18} = a\cdot i_*\bigl(\overline{\varepsilon}_3\eta_{18}\bigr) = a\cdot i_*(\eta_3\overline{\varepsilon}_4) = a\cdot(i_*\eta_3\circ\overline{\varepsilon}_4) = 0,$$

since $i_*\eta_3\in\pi_4(\mathrm{SU}(3)) = 0$. This contradicts (3.11). Therefore $[\nu_5^2]\circ\eta_{11} = 0$. □

3.7. $[C_{\eta_{11}},\mathrm{SU}(3)]_{(2)}$. By Table 1 and Proposition 3.3 (1), we have the following commutative diagram with exact rows and columns:

(3.12)

$$\begin{array}{ccccccccc}
\mathbb{Z}_4\bigl\{[\sigma''']\bigr\} & \xrightarrow{\eta_{12}^*} & \mathbb{Z}_2\{i_*\varepsilon'\} & \xrightarrow{(\Sigma^8 q_3)^*} & [C_{\eta_{11}},\mathrm{SU}(3)]_{(2)} & \xrightarrow{(\Sigma^8 i')^*} & \mathbb{Z}_4\bigl\{[\nu_5^2]\bigr\} & \longrightarrow & 0\\
\downarrow & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow & & \\
\mathbb{Z}_2\{\sigma'''\} & \xrightarrow{\eta_{12}^*} & \mathbb{Z}_2\{\varepsilon_5\} & \xrightarrow{(\Sigma^8 q_3)^*} & [C_{\eta_{11}},\mathrm{S}^5] & \xrightarrow{(\Sigma^8 i')^*} & \mathbb{Z}_2\{\nu_5^2\} & \longrightarrow & 0\\
\downarrow & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow & & \\
\mathbb{Z}_2\{\varepsilon_3\} & \xrightarrow{\eta_{11}^*} & \mathbb{Z}_2^2\{\mu_3,\eta_3\varepsilon_4\} & \xrightarrow{(\Sigma^7 q_3)^*} & [C_{\eta_{10}},\mathrm{S}^3]_{(2)} & \longrightarrow & 0 & &
\end{array}$$

The purpose of this subsection is to prove

(3.13) $$[C_{\eta_{11}},\mathrm{SU}(3)]_{(2)} = \mathbb{Z}_8\bigl\{\overline{[\nu_5^2]}\bigr\},\quad 4\cdot\overline{[\nu_5^2]} = (\Sigma^8 q_3)^* i_*\varepsilon'.$$

We need two lemmas.

Lemma 3.4. (1) $[\sigma''']\circ\eta_{12} = 0$.

(2) ([**KMNST**, Proposition 3.6]) $[C_{\eta_{11}},\mathrm{S}^5] = \mathbb{Z}_4\bigl\{p_*\overline{[\nu_5^2]}\bigr\}$, $2\cdot p_*\overline{[\nu_5^2]} = (\Sigma^8 q_3)^*\varepsilon_5$.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccc}
\pi_{12}(\mathrm{SU}(3)) & \xrightarrow[\cong]{i_{3,4*}} & \pi_{12}(\mathrm{SU}(4)) & \xrightarrow{i_{4,5*}} & \pi_{12}(\mathrm{SU}(5))\\
\downarrow{\scriptstyle\eta_{12}^*} & & \downarrow{\scriptstyle\eta_{12}^*} & & \downarrow{\scriptstyle\eta_{12}^*}\\
\pi_{13}(\mathrm{SU}(3))_{(2)} & \xrightarrow{i_{3,4*}} & \pi_{13}(\mathrm{SU}(4)) & \xrightarrow[\cong]{i_{4,5*}} & \pi_{13}(\mathrm{SU}(5))
\end{array}$$

Here $i_{k,l}:\mathrm{SU}(k)\to\mathrm{SU}(l)$ is the inclusion map. Recall from [**T1**, Theorem 4.4] that $\pi_{12}(\mathrm{SU}(5)) = \mathbb{Z}_8\oplus\mathbb{Z}_{45}$. Then the first $i_{3,4*}$ is bijective and the second $i_{3,4*}$ is injective by [**MT**]. Since $\pi_{13}(\mathrm{S}^9) = \pi_{14}(\mathrm{S}^9) = 0$ by [**T2**], the first $i_{4,5*}$ is injective and the second $i_{4,5*}$ is bijective. Let g denote a generator of the 2-primary part of $\pi_{12}(\mathrm{SU}(5))$ satisfying $i_{3,5*}[\sigma'''] = 2g$. Then

$$i_{3,5*}\eta_{12}^*[\sigma'''] = \eta_{12}^* i_{3,5*}[\sigma'''] = \eta_{12}^*(2g) = g\circ 2\eta_{12} = 0.$$

Hence $\eta_{12}^*[\sigma''']=0$ and we obtain (1).

Since no precise proof of (2) is in [**KMNST**], we give a proof of (2). We firstly claim that the second p_* of (3.12) is surjective, that is, the second ∂ of (3.12) is trivial. We have

$$\partial\varepsilon_5=\partial\iota_5\circ\varepsilon_4=\eta_3\varepsilon_4=\varepsilon_3\eta_{11}=\eta_{11}^*\varepsilon_3$$

so that

$$\partial(\Sigma^8 q_3)^*\varepsilon_5=(\Sigma^7 q_3)^*\partial\varepsilon_5=(\Sigma^7 q_3)^*\eta_{11}^*\varepsilon_3=0.$$

Of course $\partial p_*\overline{[\nu_5^2]}=0$. Hence the second ∂ of (3.12) is trivial, since $[C_{\eta_{11}},\mathrm{S}^5]$ is generated by $(\Sigma^8 q_3)^*\varepsilon_5$ and $p_*\overline{[\nu_5^2]}$.

By [**T2**, (7.4)], $\sigma'''\eta_{12}=0$. Hence, by the second row of (3.12), the order of $[C_{\eta_{11}},\mathrm{S}^5]$ is 4. To induce a contradiction, assume $[C_{\eta_{11}},\mathrm{S}^5]\cong\mathbb{Z}_2^2$, that is, $[C_{\eta_{11}},\mathrm{S}^5]=\mathbb{Z}_2^2\big\{(\Sigma^8 q_3)^*\varepsilon_5,p_*\overline{[\nu_5^2]}\big\}$. Then the surjectivity of $p_*:[C_{\eta_{11}},\mathrm{SU}(3)]_{(2)}\to[C_{\eta_{11}},\mathrm{S}^5]$ implies that $[C_{\eta_{11}},\mathrm{SU}(3)]_{(2)}$ is generated by at least two elements, that is, it must be that $[C_{\eta_{11}},\mathrm{SU}(3)]_{(2)}=\mathbb{Z}_2\big\{(\Sigma^8 q_3)^*i_*\varepsilon'\big\}\oplus\mathbb{Z}_4\big\{\overline{[\nu_5^2]}\big\}$. But this is impossible, since $p_*(\Sigma^8 q_3)^*i_*\varepsilon'=(\Sigma^8 q_3)^*p_*i_*\varepsilon'=0$. Therefore $[C_{\eta_{11}},\mathrm{S}^5]=\mathbb{Z}_4\big\{p_*\overline{[\nu_5^2]}\big\}$ with $2\cdot p_*\overline{[\nu_5^2]}=(\Sigma^8 q_3)^*\varepsilon_5$. $\square$

We use the following fibration:

$$\mathrm{SU}(3)\xrightarrow{\hat{i}}G_2\xrightarrow{\hat{p}}\mathrm{S}^6$$

We use notations and results of [**M**] freely. By [**T2**, **M**] and Table 1, we have the following commutative diagram with exact rows and columns where all groups are localized at 2:
(3.14)

$$\begin{tikzcd}
 & \mathbb{Z}_8\big\{\langle\overline{\nu}_6+\varepsilon_6\rangle\big\}\oplus\mathbb{Z}_2\big\{\hat{i}_*[\nu_5^2]\nu_{11}\big\} \arrow[r, "(\Sigma^9 q_3)^*", "\cong"'] \arrow[d, "\hat{p}_*"] & {[C_{\eta_{12}},G_2]} \arrow[d, "\hat{p}_*"] & \\
\mathbb{Z}_4\{\sigma''\} \arrow[r, "\eta_{13}^*"] \arrow[d, "\partial"] & \mathbb{Z}_8\{\overline{\nu}_6\}\oplus\mathbb{Z}_2\{\varepsilon_6\} \arrow[r, "(\Sigma^9 q_3)^*"] \arrow[d, "\partial"] & {[C_{\eta_{12}},\mathrm{S}^6]} \arrow[r, two heads, "(\Sigma^9 i')^*"] \arrow[d, "\partial"] & \mathbb{Z}_2\{\nu_6^2\} \arrow[d, "\partial"] \\
\mathbb{Z}_4\big\{[\sigma''']\big\} \arrow[r, "\eta_{12}^*=0"] & \mathbb{Z}_2\{i_*\varepsilon'\} \arrow[r, "(\Sigma^8 q_2)^*"] \arrow[d] & {[C_{\eta_{11}},\mathrm{SU}(3)]} \arrow[r, two heads, "(\Sigma^8 i')^*"] \arrow[d, "\hat{i}_*"] & \mathbb{Z}_4\big\{\overline{[\nu_5^2]}\big\} \arrow[d, "\hat{i}_*"] \\
 & 0 \arrow[r] & {[C_{\eta_{11}},G_2]} \arrow[r, "(\Sigma^8 i')^*", "\cong"'] & \mathbb{Z}_2\big\{\hat{i}_*\overline{[\nu_5^2]}\big\}\oplus\mathbb{Z}_{(2)}
\end{tikzcd}$$

Here we have used results of [**M**] that $\pi_{12}(G_2)=\pi_{13}(G_2)=0$. We need

Lemma 3.5. (1) ([**M**, Proposition 6.3]) $\partial\overline{\nu}_6=\partial\varepsilon_6=i_*\varepsilon'$.

(2) ([**KMNST**, Proposition 3.6]) $[C_{\eta_{12}},\mathrm{S}^6]_{(2)}=\mathbb{Z}_4\big\{(\Sigma^9 q_3)^*\overline{\nu}_6\big\}\oplus\mathbb{Z}_4\big\{\Sigma p_*\overline{[\nu_5^2]}\big\}$ *and* $2\cdot\Sigma p_*\overline{[\nu_5^2]}=(\Sigma^9 q_3)^*\varepsilon_6$.

Proof. We give a proof of (2), because our notations are different from ones in [**KMNST**]. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\mathbb{Z}_2\{\sigma'''\} & \xrightarrow{\eta_{12}^*=0} & \mathbb{Z}_2\{\varepsilon_5\} & \xrightarrow{(\Sigma^8 q_3)^*} & [C_{\eta_{11}}, \mathrm{S}^5]_{(2)} & \xrightarrow{(\Sigma^8 i')^*} & \mathbb{Z}_2\{\nu_5^2\} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} & & \cong\downarrow{\scriptstyle\Sigma} & & \\
\mathbb{Z}_4\{\sigma''\} & \xrightarrow{\eta_{13}^*} & \mathbb{Z}_8\{\overline{\nu}_6\} \oplus \mathbb{Z}_2\{\varepsilon_6\} & \xrightarrow{(\Sigma^9 q_3)^*} & [C_{\eta_{12}}, \mathrm{S}^6]_{(2)} & \xrightarrow{(\Sigma^9 i')^*} & \mathbb{Z}_2\{\nu_6^2\} & \longrightarrow & 0
\end{array}$$

By Lemma 3.4 (2), we have

$$2\Sigma p_*\overline{[\nu_5^2]} = (\Sigma^9 q_3)^*\Sigma\varepsilon_5 = (\Sigma^9 q_3)^*\varepsilon_6. \tag{3.15}$$

We have $\eta_{13}^*\sigma'' = 4\cdot\overline{\nu}_6$ by [**T2**, (7.4)] so that we have the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}_4\big\{(\Sigma^9 q_3)^*\overline{\nu}_6\big\} \oplus \mathbb{Z}_2\big\{(\Sigma^9 q_3)^*\varepsilon_6\big\} \longrightarrow [C_{\eta_{12}}, \mathrm{S}^6]_{(2)} \xrightarrow{(\Sigma^9 i')^*} \mathbb{Z}_2\{\nu_6^2\} \longrightarrow 0$$

Thus the order of $\Sigma p_*\overline{[\nu_5^2]}$ is 4 by (3.15), and we obtain (2) by the above exact sequence, since $(\Sigma^9 i')^*\Sigma p_*\overline{[\nu_5^2]} = \nu_6^2$. □

Proof of (3.13). We have

$$0 = \partial\hat{p}_*(\Sigma^9 q_3)^*\langle\overline{\nu}_6 + \varepsilon_6\rangle = \partial(\Sigma^9 q_3)^*(\overline{\nu}_6 + \varepsilon_6) = \partial(\Sigma^9 q_3)^*\overline{\nu}_6 + 2\cdot\partial\Sigma p_*\overline{[\nu_5^2]},$$

where the last equality follows from (3.15). Hence

$$-2\cdot\partial\Sigma p_*\overline{[\nu_5^2]} = \partial(\Sigma^9 q_3)^*\overline{\nu}_6 = (\Sigma^8 q_3)^*\partial\overline{\nu}_6 = (\Sigma^8 q_3)^* i_*\varepsilon',$$

where the last equality follows from Lemma 3.5 (1). Thus the order of $\partial\Sigma p_*\overline{[\nu_5^2]}$ is 4. On the other hand,

$$(\Sigma^8 i')^*(2\cdot\overline{[\nu_5^2]}) = 2[\nu_5^2] = \partial\nu_6^2 = \partial(\Sigma^9 i')^*\Sigma p_*\overline{[\nu_5^2]} = (\Sigma^8 i')^*\partial\Sigma p_*\overline{[\nu_5^2]}.$$

Hence there exists an integer x such that $2\cdot\overline{[\nu_5^2]} - \partial\Sigma p_*\overline{[\nu_5^2]} = x\cdot(\Sigma^8 q_3)^* i_*\varepsilon'$. Thus $4\cdot\overline{[\nu_5^2]} = 2\cdot\partial\Sigma p_*\overline{[\nu_5^2]} = (\Sigma^8 q_3)^* i_*\varepsilon'$. Therefore the order of $\overline{[\nu_5^2]}$ is 8, and we obtain (3.13).

4. $\pi_n\mathrm{map}_*(\mathrm{Sp}(2), \mathrm{Sp}(2))$

In this section we compute $\pi_n\mathrm{map}_*(\mathrm{Sp}(2), \mathrm{Sp}(2))$. Let $f : \mathrm{S}^9 \to \mathrm{S}^3 \cup_\omega e^7$ be the attaching map of the top cell of Sp(2), that is, $\mathrm{Sp}(2) = \mathrm{S}^3 \cup_\omega e^7 \cup_f e^{10}$. The double suspension of f is trivial, that is $\Sigma^2 f = 0$, because $\Sigma^2 f$ is an element of the homotopy group $\pi_{11}(\mathrm{S}^5 \cup_{\Sigma^2\omega} e^9)$ which is isomorphic to the stable group, while f is a stably trivial element by [**BS**]. Thus we obtain

$$\Sigma^2\,\mathrm{Sp}(2) \simeq \mathrm{S}^5 \cup_{\Sigma^2\omega} e^9 \vee \mathrm{S}^{12}\,.$$

The p-components of the homotopy groups for $p \geq 5$ are easily obtained from the results in [**T2**], since if $p \geq 5$

$$\mathrm{Sp}(2)_{(p)} \simeq \mathrm{S}^3_{(p)} \times \mathrm{S}^7_{(p)}$$

and thus for $n \geq 1$

$$[\Sigma^n\,\mathrm{Sp}(2), \mathrm{Sp}(2)]_{(p)} \cong \big(\pi_{n+3}(\mathrm{S}^3 \times \mathrm{S}^7) \oplus \pi_{n+7}(\mathrm{S}^3 \times \mathrm{S}^7) \oplus \pi_{n+10}(\mathrm{S}^3 \times \mathrm{S}^7)\big)_{(p)}. \tag{4.1}$$

Hence we must compute 2 and 3 components of $[\Sigma^n \mathrm{Sp}(2), \mathrm{Sp}(2)]$ for $n \geq 1$. The following table shows the generators of 2 and 3 components. Here we use the same notation as before.

n	$2,3$-*components*	*generators*
1	$\mathbb{Z}_2^2$	$\Sigma q^* i_*\varepsilon_3,\ \overline{\overline{i_*\eta_3}}$
2	$\mathbb{Z}_2^3$	$\Sigma^2 q^* i_*\mu_3,\ \Sigma^2 q^* i_*(\eta_3\varepsilon_3),\ \overline{\overline{i_*\eta_3^2}}$
3	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$	$\Sigma^3 q^* i_*(\eta_3\mu_4),\ \Sigma^3 q^*([\nu_7]\nu_{10}),\ \overline{\Sigma^3 {q_3}^*[\nu_7]}$
4	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_3$	$\overline{\overline{3[12\iota_7]}},\ \overline{\Sigma^4 {q_3}^* i_*\varepsilon_3},\ \Sigma^4 q^*[2\sigma'],\ \Sigma^4 q^* i_*\alpha_3(3)$
5	$\mathbb{Z}_2^3$	$\Sigma^5 q^*[\sigma'\eta_{14}],\ \overline{\Sigma^5 {q_3}^* i_*\mu_3},\ \overline{\Sigma^5 {q_3}^* i_*(\eta_3\varepsilon_4)}$
6	$\mathbb{Z}_2^4$	$\Sigma^6 q^*([\sigma'\eta_{14}] \circ \eta_{15}),\ \Sigma^6 q^*([\nu_7] \circ \nu_{10}^2),\ \overline{\Sigma^6 {q_3}^*([\nu_7] \circ \nu_{10})},\ \overline{\Sigma^6 {q_3}^* i_*(\eta_3\mu_4)}$
7	$\mathbb{Z}_8 \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9$	$\Sigma^7 q^*([\nu_7] \circ \sigma_{10}),\ \overline{\overline{2[\nu_7]}},\ 2 \cdot \overline{\overline{2[\nu_7]}} - z \cdot \overline{\Sigma^7 {q_3}^*[2\sigma']},\ \overline{\overline{i_*\alpha_2(3)}}$
8	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_9$	$\Sigma^8 q^* i_*\bar{\varepsilon}_3, \overline{\overline{i_*\varepsilon_3}}, \overline{\Sigma^8 {q_3}^*[\sigma'\eta_{14}]}, \Sigma^8 q^*[\zeta_7], \Sigma^8 q^*[\alpha'_3(7)]$

Table 3: 2 and 3 components of $\pi_n \mathrm{map}_*(\mathrm{Sp}(2), \mathrm{Sp}(2))$

Here z is an odd integer.

As in the SU(3) case, we obtain the following lemma.

Lemma 4.1. $[\Sigma^n \mathrm{Sp}(2), \mathrm{Sp}(2)] \cong \pi_{10+n}(\mathrm{Sp}(2)) \oplus [C_{\Sigma^n\omega}, \mathrm{Sp}(2)]$ *for* $n \geq 1$.

Proof. The proof is similar to that of Lemma 3.2. □

Hence it suffices for our purpose to determine $[C_{\Sigma^n\omega}, \mathrm{Sp}(2)]_{(2,3)}$, the 2 and 3 components of $[C_{\Sigma^n\omega}, \mathrm{Sp}(2)]$, for $n \geq 1$. We use the following results of Mimura-Toda [**MT**].

n	$\pi_n \mathrm{Sp}(2)$	gen. of $2,3$-comp.	n	$\pi_n \mathrm{Sp}(2)$	gen. of $2,3$-comp.
$1,2,6,8,9$	0		12	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$i_*\mu_3,\ i_*\eta_3\varepsilon_3$
3	$\mathbb{Z}$	$i_*\iota_3$	13	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$[\nu_7] \circ \nu_{10},\ i_*\eta_3\mu_4$
4	$\mathbb{Z}_2$	$i_*\eta_3$	14	$\mathbb{Z}_{16} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{35}$	$[2\sigma'],\ i_*\alpha_3(3)$
5	$\mathbb{Z}_2$	$i_*\eta_3^2$	15	$\mathbb{Z}_2$	$[\sigma'\eta_{14}]$
7	$\mathbb{Z}$	$[12\iota_7]$	16	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$[\sigma'\eta_{14}] \circ \eta_{15},\ [\nu_7] \circ \nu_{10}^2$
10	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	$[\nu_7],\ i_*\alpha_2(3)$	17	$\mathbb{Z}_8 \oplus \mathbb{Z}_5$	$[\nu_7] \circ \sigma_{10}$
11	$\mathbb{Z}_2$	$i_*\varepsilon_3$	18	$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{35}$	$[\zeta_7], i_*\overline{\varepsilon}_3,\ [3 \cdot \alpha'_3(7)]$

Table 4 : $\pi_n(\mathrm{Sp}(2))$

4.1. $[C_{\Sigma^n\omega},\mathrm{Sp}(2)]$ $(n=1,2)$**.** By the cofibration sequence and Table 4, it is easy to see that

$$[C_{\Sigma\omega},\mathrm{Sp}(2)]=\mathbb{Z}_2\big\{i_*\overline{\eta_3}\big\},\quad [C_{\Sigma^2\omega},\mathrm{Sp}(2)]=\mathbb{Z}_2\big\{i_*\big(\eta_3\circ\Sigma\overline{\eta_3}\big)\big\}.$$

4.2. $[C_{\Sigma^3\omega},\mathrm{Sp}(2)]$**.** By Table 4, we have the following exact sequence.

$$\mathbb{Z}\big\{[12\iota_7]\big\}\xrightarrow{(\Sigma^4\omega)^*}\mathbb{Z}_8\big\{[\nu_7]\big\}\oplus\mathbb{Z}_3\{i_*\alpha_2(3)\}\oplus\mathbb{Z}_5\longrightarrow[C_{\Sigma^3\omega},\mathrm{Sp}(2)]\longrightarrow 0. \tag{4.2}$$

Lemma 4.2. $(\Sigma^4\omega)^*[12\iota_7]=i_*\alpha_2(3)$.

Proof. It is known that $\Sigma^4\omega=2\nu_7+\alpha_1(7)$. Let $p:\mathrm{Sp}(2)\to\mathrm{S}^7$ be the bundle projection with fibre S^3. Then $p_*([12\iota_7]\circ 2\nu_7)=0$, and hence $[12\iota_7]\circ 2\nu_7=0$ by Table 4. Next consider the composition $[12\iota_7]\circ\alpha_1(7)$. We apply Theorem 3.1 to the fibration $p:\mathrm{Sp}(2)\to\mathrm{S}^7$ by taking $\alpha=4\iota_7,\ \beta=3\iota_6,\ \gamma=\alpha_1(6)$. Then we obtain

$$[12\iota_7]\circ\alpha_1(7)=i_*\alpha_2(3). \tag{4.3}$$

Hence $(\Sigma^4\omega)^*[12\iota_7]=i_*\alpha_2(3)$ as desired. □

Consequently, by (4.2) we obtain

$$[C_{\Sigma^3\omega},\mathrm{Sp}(2)]_{(2,3)}=\mathbb{Z}_8\big\{(\Sigma^3q_3)^*[\nu_7]\big\}.$$

4.3. $[C_{\Sigma^4\omega},\mathrm{Sp}(2)]$**.** By Table 4, we have the following exact sequence.

$$0\longrightarrow\mathbb{Z}_2\{i_*\varepsilon_3\}\xrightarrow{(\Sigma^4q_3)^*}[C_{\Sigma^4\omega},\mathrm{Sp}(2)]\xrightarrow{(\Sigma^4i')^*}\mathbb{Z}\big\{[12\iota_7]\big\}\xrightarrow{\Sigma^4\omega^*}\mathbb{Z}_{120}$$

By Lemma 4.2, $\mathrm{Ker}(\Sigma^4\omega)^*=\mathbb{Z}\big\{3[12\iota_7]\big\}$. It follows that

$$[C_{\Sigma^4\omega},\mathrm{Sp}(2)]=\mathbb{Z}_2\big\{(\Sigma^4q_3)^*i_*\varepsilon_3\big\}\oplus\mathbb{Z}\big\{\overline{3[12\iota_7]}\big\}.$$

4.4. $[C_{\Sigma^5\omega},\mathrm{Sp}(2)]$**.** By Table 4, we easily have $(\Sigma^5q_3)^*:\pi_{12}(\mathrm{Sp}(2))\cong[C_{\Sigma^5\omega},\mathrm{Sp}(2)]$. Hence

$$[C_{\Sigma^5\omega},\mathrm{Sp}(2)]=\mathbb{Z}_2\big\{(\Sigma^5q_3)^*i_*\mu_3\big\}\oplus\mathbb{Z}_2\big\{(\Sigma^5q_3)^*i_*(\eta_3\varepsilon_4)\big\}.$$

4.5. $[C_{\Sigma^6\omega},\mathrm{Sp}(2)]$**.** By Table 4, we have the following exact sequence.

$$\mathbb{Z}_8\big\{[\nu_7]\big\}\oplus\mathbb{Z}_{15}\xrightarrow{(\Sigma^7\omega)^*}\mathbb{Z}_4\big\{[\nu_7]\circ\nu_{10}\big\}\oplus\mathbb{Z}_2\{i_*\eta_3\mu_4\}\xrightarrow{(\Sigma^6q_3)^*}[C_{\Sigma^6\omega},\mathrm{Sp}(2)]\longrightarrow 0$$

Hence we obtain

$$[C_{\Sigma^6\omega},\mathrm{Sp}(2)]=\mathbb{Z}_2\big\{(\Sigma^6q_3)^*[\nu_7]\circ\nu_{10}\big\}\oplus\mathbb{Z}_2\big\{(\Sigma^6q_3)^*i_*(\eta_3\mu_4)\big\}.$$

4.6. $[C_{\Sigma^7\omega}, \mathrm{Sp}(2)]$**.** By Table 4, we have the following exact sequence:

$$0 \to \mathbb{Z}_{16}\big\{[2\sigma']\big\} \oplus \mathbb{Z}_3\{i_*\alpha_3(3)\} \overset{(\Sigma^7 q_3)^*}{\to} [C_{\Sigma^7\omega}, \mathrm{Sp}(2)]_{(2,3)} \overset{(\Sigma^7 i')^*}{\to} \mathbb{Z}_4\big\{2[\nu_7]\big\} \oplus \mathbb{Z}_3\{i_*\alpha_2(3)\} \to 0.$$

We shall prove

$$[C_{\Sigma^7\omega}, \mathrm{Sp}(2)]_{(2)} = \mathbb{Z}_{32}\Big\{\overline{2[\nu_7]}\Big\} \oplus \mathbb{Z}_2\Big\{2\cdot\overline{2[\nu_7]} - z\cdot(\Sigma^7 q_3)^*[2\sigma']\Big\},\ z \equiv 1 \pmod 2, \tag{4.4}$$

$$[C_{\Sigma^7\omega}, \mathrm{Sp}(2)]_{(3)} = \mathbb{Z}_9\big\{\overline{i_*\alpha_2(3)}\big\}. \tag{4.5}$$

Firstly we prove (4.4). By Table 4 and [**T2**], we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & \mathbb{Z}_{16}\big\{[2\sigma']\big\} \arrow[r, tail, "q^*"] \arrow[d, "p_*"] & {[C_{\Sigma^7\omega}, \mathrm{Sp}(2)]_{(2)}} \arrow[r, two heads, "i^*"] \arrow[d, "p_*"] & \mathbb{Z}_4\big\{2[\nu_7]\big\} \arrow[d, "p_*"] \\
 & \mathbb{Z}_8\{\sigma'\} \arrow[r, tail, "q^*"] \arrow[d, "\partial"] & {[C_{\Sigma^7\omega}, \mathrm{S}^7]_{(2)}} \arrow[r, two heads, "i^*"] \arrow[d, "\partial"] & \mathbb{Z}_8\{\nu_7\} \\
 & \mathbb{Z}_4\{\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \arrow[r, tail, two heads, "q^*"] \arrow[d, "i_*"] & {[C_{\Sigma^6\omega}, \mathrm{S}^3]_{(2)}} \arrow[d, "i_*"] & \\
\mathbb{Z}_8\big\{[\nu_7]\big\} \arrow[r, "(2\nu_{10})^*"] & \mathbb{Z}_4\big\{[\nu_7]\circ\nu_{10}\big\} \oplus \mathbb{Z}_2\{i_*\eta_3\mu_4\} \arrow[r, "q^*"] & {[C_{\Sigma^6\omega}, \mathrm{Sp}(2)]_{(2)}} &
\end{tikzcd} \tag{4.6}$$

We claim that the second row splits:

$$[C_{\Sigma^7\omega}, \mathrm{S}^7]_{(2)} = \mathbb{Z}_8\{q^*\sigma'\} \oplus \mathbb{Z}_8\{\overline{\nu_7}\}. \tag{4.7}$$

This is done as follows. By [**T2**], we easily have

$$[C_{\Sigma^3\omega}, \mathrm{S}^3]_{(2)} = \mathbb{Z}_4\big\{\overline{\nu'}\big\} \tag{4.8}$$

and the following exact sequence:

$$0 \longrightarrow \mathbb{Z}_2\{\sigma'''\} \xrightarrow{q^*} [C_{\Sigma^5\omega}, \mathrm{S}^5]_{(2)} \xrightarrow{i^*} \mathbb{Z}_8\{\nu_5\} \longrightarrow 0.$$

Since $i^*\big(2\cdot\overline{\nu_5} - \Sigma^2\overline{\nu'}\big) = 0$, we can write $2\cdot\overline{\nu_5} - \Sigma^2\overline{\nu'} = c\cdot q^*\sigma'''$ ($c \in \mathbb{Z}$). Then $4\cdot\overline{\nu_5} - 2\cdot\Sigma^2\overline{\nu'} = 0$ so that the order of $\overline{\nu_5}$ is 8, since $i^*(2\cdot\Sigma^2\overline{\nu'}) = 4\nu_5$ so that the order of $2\cdot\Sigma^2\overline{\nu'}$ is 2 by (4.8). Define $\overline{\nu_7} := \Sigma^2\overline{\nu_5}$. Then the order of $\overline{\nu_7}$ is 8, for the order of $i^*(\overline{\nu_7}) = \nu_7$ is 8. Thus we obtain (4.7).

In (4.6), we have $i^*\varepsilon' = 2[\nu_7]\circ\nu_{10} = (\Sigma^7\omega)^*[\nu_7]$ by [**MT**]. Hence $\partial\sigma' = 2\varepsilon'$, $i_*q^*\varepsilon' = q^*i_*\varepsilon' = 0$ and

$$\partial q^*\sigma' = q^*\partial\sigma' = 2q^*\varepsilon'. \tag{4.9}$$

Hence the kernel of the second i_* of (4.6) equals to $\mathbb{Z}_4\{q^*\varepsilon'\}$. This kernel equals to the image of the second ∂ of (4.6). Hence

$$\partial\overline{\nu_7} = \pm q^*\varepsilon' \tag{4.10}$$

by (4.7) and (4.9). We have $i^*\big(2\cdot\overline{\nu_7} - p_*\overline{2[\nu_7]}\big) = 0$ so that we can write

$$2\cdot\overline{\nu_7} - p_*\overline{2[\nu_7]} = a\cdot q^*\sigma' \quad (a \in \mathbb{Z}). \tag{4.11}$$

We then have

$$\begin{aligned} 2a \cdot q^*\varepsilon' &= \partial(a \cdot q^*\sigma') \quad (\text{by } (4.9)) \\ &= \partial\big(2 \cdot \overline{\nu_7} - p_*\overline{2[\nu_7]}\big) = 2 \cdot \partial\overline{\nu_7} \\ &= 2 \cdot q^*\varepsilon' \quad (\text{by } (4.10). \end{aligned}$$

Hence $2a \equiv 2 \pmod 4$, that is, a is odd. It follows that, by multiplying 4 with (4.11), we have

$$4 \cdot q^*\sigma' = -4 \cdot p_*\overline{2[\nu_7]}.$$

On the other hand, we can write

$$4 \cdot \overline{2[\nu_7]} = y \cdot q^*[2\sigma'] \quad (y \in \mathbb{Z}). \tag{4.12}$$

Hence we have

$$4 \cdot q^*\sigma' = -y \cdot p_*q^*[2\sigma'] = -2y \cdot q^*\sigma'.$$

Hence $-2y \equiv 4 \pmod 8$, that is,

$$y \equiv 2 \pmod 4. \tag{4.13}$$

Thus the order of $4 \cdot \overline{2[\nu_7]}$ is 8, that is, the order of $\overline{2[\nu_7]}$ is 32. Also the order of $2 \cdot \overline{2[\nu_7]} - (y/2) \cdot q^*[2\sigma']$ is 2. Therefore we obtain (4.4) by the first row of (4.6).

As a byproduct of (4.13), we have

Corollary 4.3. $[\nu_7] \circ \eta_{10} = i_*\varepsilon_3 \in \pi_{11}(\mathrm{Sp}(2)) = \mathbb{Z}_2\{i_*\varepsilon_3\}$.

Proof. Since indeterminacy of $\{2[\nu_7], 2\nu_{10}, 4\iota_{13}\}$ is $4 \cdot \pi_{14}(\mathrm{Sp}(2))$, we can write

$$\big\{2[\nu_7], 2\nu_{10}, 4\iota_{13}\big\} = x \cdot [2\sigma'] + 4 \cdot \pi_{14}(\mathrm{Sp}(2)). \tag{4.14}$$

Let $\psi^k : \mathrm{Sp}(2) \to \mathrm{Sp}(2)$ be defined by $\psi^k(A) = A^k$. We have

$$\psi^2 \circ \big\{2[\nu_7], 2\nu_{10}, 4\iota_{13}\big\} \subset \big\{4[\nu_7], 2\nu_{10}, 4\iota_{13}\big\} \subset \big\{[\nu_7], 8\nu_{10}, 4\iota_{13}\big\} = 4\pi_{14}(\mathrm{Sp}(2)).$$

Hence $2x[2\sigma'] \in 4\pi_{14}(\mathrm{Sp}(2)) = \mathbb{Z}_4\big\{4[2\sigma']\big\} \oplus \mathbb{Z}_{105}$ by Table 4. Thus $x \equiv 0 \pmod 2$. On the other hand

$$\big\{2[\nu_7], 2\nu_{10}, 4\iota_{13}\big\} = \big\{[\nu_7], 4\nu_{10}, 4\iota_{13}\big\} = \big\{[\nu_7], \eta_{10}^3, 4\iota_{13}\big\} = \{[\nu_7] \circ \eta_{10}, \eta_{11}^2, 4\iota_{13}\}. \tag{4.15}$$

To induce a contradiction, assume $[\nu_7] \circ \eta_{10} = 0$. Then $\big\{2[\nu_7], 2\nu_{10}, 4\iota_{13}\big\} = 4\pi_{14}(\mathrm{Sp}(2))$ by (4.15) and $x \equiv 0 \pmod 4$ by (4.14). We then have

$$\begin{aligned} 0 &= 4 \cdot \big(\overline{2[\nu_7]} \circ \widetilde{4\iota_{13}}\big) = \psi^4 \circ \overline{2[\nu_7]} \circ \widetilde{4\iota_{13}} = \big(4 \cdot \overline{2[\nu_7]}\big) \circ \widetilde{4\iota_{13}} \\ &= \big(y \cdot q^*[2\sigma']\big) \circ \widetilde{4\iota_{13}} \quad (\text{by } (4.12)) \\ &= \psi^y \circ [2\sigma'] \circ q \circ \widetilde{4\iota_{13}} = \psi^y \circ [2\sigma'] \circ 4\iota_{14} \\ &= 4y[2\sigma'] \end{aligned}$$

Thus $4y \equiv 0 \pmod{16}$, that is, $y \equiv 0 \pmod 4$. This contradicts (4.13). ∎

Next we consider the 3-primary part of $[C_{\Sigma^7\omega}, \mathrm{Sp}(2)]$, that is, we prove (4.5). First we remark that
$$[C_{\Sigma^7\omega}, \mathrm{Sp}(2)]_{(3)} \cong [C_{\alpha_1(10)}, \mathrm{Sp}(2)]_{(3)}.$$
Hence it suffices to prove
$$[C_{\alpha_1(10)}, \mathrm{Sp}(2)]_{(3)} \cong \mathbb{Z}_9.$$
We shall prove this as follows.

Proposition 4.4. (1) $\{\alpha_1(5), \alpha_1(8), 3\iota_{11}\} = \{3\iota_5, \alpha_1(5), \alpha_1(8)\} = 2\alpha_2(5) + 3\pi_{12}(\mathrm{S}^5)$.

(2) $i \circ \alpha_3(3) = [12\iota_7] \circ \alpha_2(7) \in \pi_{10}(\mathrm{Sp}(2))$.

(3) $[C_{\alpha_1(10)}, \mathrm{Sp}(2)]_{(3)} \cong [C_{\alpha_1(10)}, \mathrm{S}^7]_{(3)}$.

(4) $[C_{\alpha_1(10)}, \mathrm{S}^7]_{(3)} \cong \mathbb{Z}_9$.

Proof of Proposition 4.4 (1). It follows from [**T2**, Proposition 1.3] that
$$\Sigma^\infty\{\alpha_1(5), \alpha_1(8), 3\iota_{11}\} \subset \langle \alpha_1, \alpha_1, 3\rangle, \quad \Sigma^\infty\{3\iota_5, \alpha_1(5), \alpha_1(8)\} \subset \langle 3, \alpha_1, \alpha_1\rangle,$$
$$\Sigma^\infty\{\alpha_1(3), 3\iota_6, \alpha_2(6)\} \subset \langle \alpha_1, 3, \alpha_2\rangle.$$
We use following relations [**T2**, (3.9)]:
$$\begin{aligned}&\langle \alpha_1, \alpha_1, 3\rangle - \langle \alpha_1, 3, \alpha_1\rangle + \langle 3, \alpha_1, \alpha_1\rangle \ni 0,\\ &\langle \alpha_1, \alpha_1, 3\rangle = \langle 3, \alpha_1, \alpha_1\rangle.\end{aligned} \tag{4.16}$$
Let $A \in \langle \alpha_1, \alpha_1, 3\rangle$. Since $\langle \alpha_1, 3, \alpha_1\rangle = \alpha_2$ and $\mathrm{Indet}\langle \alpha_1, \alpha_1, 3\rangle = 3G_7$, it follows from (4.16) that $2A - \alpha_2 + 3G_7 \ni 0$ so that $A \in 2\alpha_2 + 3G_7$, since $G_{7(3)} = \mathbb{Z}_3\{\alpha_2\}$, where G_k denotes the k-th stable homotopy group of the sphere. Hence $\langle \alpha_1, \alpha_1, 3\rangle = 2\alpha_2 + 3G_7$. Since $\Sigma^\infty : \pi_{12}(\mathrm{S}^5) = \mathbb{Z}_3\{\alpha_2(5)\} \oplus \mathbb{Z}_{10} \to G_7$ is injective and $\mathrm{Indet}\{\alpha_1(5), \alpha_1(8), 3\iota_{11}\} = \mathrm{Indet}\{3\iota_5, \alpha_1(5), \alpha_1(8)\} = 3\pi_{12}(\mathrm{S}^5)$, it follows that
$$2\alpha_2(5) \in \{\alpha_1(5), \alpha_1(8), 3\iota_{11}\} \cap \{3\iota_5, \alpha_1(5), \alpha_1(8)\}$$
so that $\{\alpha_1(5), \alpha_1(8), 3\iota_{11}\} = \{3\iota_5, \alpha_1(5), \alpha_1(8)\} = 2\alpha_2(5) + 3\pi_{12}(\mathrm{S}^5)$. □

Proof of Proposition 4.4 (2). We can apply Theorem 3.1 to the fibration $\mathrm{Sp}(2) \to \mathrm{S}^7$ by taking $\alpha = 4\iota_7$, $\beta = 3\iota_6$ and $\gamma = \alpha_2(6)$. Indeed, we have $\beta \circ \gamma = 0$ and $\partial\alpha \circ \beta = \alpha_1(3) \circ 3\iota_6 = 0$ since $\partial\iota_7 = \omega = \nu' + \alpha_1(3)$. Hence we can use Theorem 3.1 in this case. Therefore there exists $\epsilon \in \pi_7(\mathrm{Sp}(2))$ such that $p_*\epsilon = 12\iota_7$ and $i_*(\alpha_3(3)) = \epsilon \circ \alpha_2(7)$ so that $\epsilon = [12\iota_7]$ and $i_*(\alpha_3(3)) = [12\iota_7] \circ \alpha_2(7)$. □

Proof of Proposition 4.4 (3). By [**T2**] and Table 4, we have the following commutative diagram with exact rows.
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_3\{i_*\alpha_3(3)\} & \longrightarrow & [C_{\alpha_1(10)}, \mathrm{Sp}(2)]_{(3)} & \longrightarrow & \mathbb{Z}_3\{i_*\alpha_2(3)\} & \longrightarrow & 0\\ & & \Big\uparrow{\scriptstyle [12\iota_7]_*} & & \Big\uparrow{\scriptstyle [12\iota_7]_*} & & \Big\uparrow{\scriptstyle [12\iota_7]_*} & & \\ 0 & \longrightarrow & \mathbb{Z}_3\{\alpha_2(7)\} & \longrightarrow & [C_{\alpha_1(10)}, \mathrm{S}^7]_{(3)} & \longrightarrow & \mathbb{Z}_3\{\alpha_1(7)\} & \longrightarrow & 0.\end{array}$$
It follows from (4.3) and Proposition 4.4 (2) that the first and the third $[12\iota_7]_*$ are isomorphisms so that the second $[12\iota_7]_*$ is also an isomorphism. Hence we obtain Proposition 4.4 (3). □

Proof of Proposition 4.4 (4). We shall prove the following:
$$[C_{\alpha_1(10)}, \mathrm{S}^7]_{(3)} \overset{\Sigma}{\cong} [C_{\alpha_1(9)}, \mathrm{S}^6]_{(3)} \overset{\Sigma}{\cong} [C_{\alpha_1(8)}, \mathrm{S}^5]_{(3)} = \mathbb{Z}_9\{\overline{\alpha_1(5)}\}.$$

By [**T2**] and the fact $\alpha_1(5)\circ\alpha_1(8)=0$ ([**T2**, (13.7)]), we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_3\{\alpha_2(5)\}\oplus\mathbb{Z}_{10} & \xrightarrow{\Sigma^5 q'^*} & [C_{\alpha_1(8)},\mathrm{S}^5] & \xrightarrow{\Sigma^5 i''^*} & \mathbb{Z}_3\{\alpha_1(5)\}\oplus\mathbb{Z}_8 & \longrightarrow & 0\\
 & & \big\downarrow\Sigma & & \big\downarrow\Sigma & & \cong\big\downarrow\Sigma & & \\
0 & \longrightarrow & \mathbb{Z}_3\{\alpha_2(6)\}\oplus\mathbb{Z}_{20} & \xrightarrow{\Sigma^6 q'^*} & [C_{\alpha_1(9)},\mathrm{S}^6] & \xrightarrow{\Sigma^6 i''^*} & \mathbb{Z}_3\{\alpha_1(6)\}\oplus\mathbb{Z}_8 & \longrightarrow & 0\\
 & & \big\downarrow\Sigma & & \big\downarrow\Sigma & & \cong\big\downarrow\Sigma & & \\
0 & \longrightarrow & \mathbb{Z}_3\{\alpha_2(7)\}\oplus\mathbb{Z}_{40} & \xrightarrow{\Sigma^7 q'^*} & [C_{\alpha_1(10)},\mathrm{S}^7] & \xrightarrow{\Sigma^7 i''^*} & \mathbb{Z}_3\{\alpha_1(7)\}\oplus\mathbb{Z}_8 & \longrightarrow & 0
\end{array}$$

Here $q' : C_{\alpha_1(3)}\to \mathrm{S}^7$ is the quotient and $i'' : \mathrm{S}^3\to C_{\alpha_1(3)}$ is the inclusion. By the EHP-sequence ([**T2**, (2.11)]), we know that two Σ's in the first column are monomorphisms. Hence two Σ's in the second column are also monomorphisms. Thus suspensions induce

$$[C_{\alpha_1(8)},\mathrm{S}^5]_{(3)}\cong[C_{\alpha_1(9)},\mathrm{S}^6]_{(3)}\cong[C_{\alpha_1(10)},\mathrm{S}^7]_{(3)}.$$

Since $\Sigma\big(3\,\overline{\alpha_1(5)}\big)=\Sigma\big(3\iota_5\circ\overline{\alpha_1(5)}\big)$, it follows that $3\,\overline{\alpha_1(5)}=3\iota_5\circ\overline{\alpha_1(5)}$. We have

$$\begin{aligned}
3\iota_5\circ\overline{\alpha_1(5)} &\in \{3\iota_5,\alpha_1(5),\alpha_1(8)\}\circ\Sigma^5 q' &&\text{(by [\textbf{T2}, Proposition 1.9])}\\
&= \big(2\alpha_2(5)+3\pi_{12}(\mathrm{S}^5)\big)\circ\Sigma^5 q' &&\text{(by Proposition 4.4 (1))}
\end{aligned}$$

Hence we can write

$$3\,\overline{\alpha_1(5)}=3\iota_5\circ\overline{\alpha_1(5)}=\Sigma^5 q'^*\big(2\alpha_2(5)+x\big),\quad 10x=0.$$

Thus the order of $\overline{\alpha_1(5)}$ is a multiple of 9. Therefore $[C_{\alpha_1(8)},\mathrm{S}^5]_{(3)}=\mathbb{Z}_9\big\{\overline{\alpha_1(5)}\big\}$. This completes the proof of Proposition 4.4. $\square$

4.7. $[C_{\Sigma^8\omega},\mathrm{Sp}(2)]$. Since $\Sigma^m\omega=2\nu_{m+3}+\alpha_1(m+3)$ for $m\geq 2$, we have

$$(\Sigma^9\omega)^*\pi_{12}(\mathrm{Sp}(2))=0,\quad (\Sigma^8\omega)^*\pi_{11}(\mathrm{Sp}(2))=0$$

by Table 4. Hence we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_2\big\{[\sigma'\eta_{14}]\big\} & \xrightarrow{(\Sigma^8 q_3)^*} & [C_{\Sigma^8\omega},\mathrm{Sp}(2)] & \xrightarrow{(\Sigma^8 i')^*} & \mathbb{Z}_2\{i_*\varepsilon_3\} & \longrightarrow & 0\\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_2\{\sigma'\eta_{14}\}\oplus\mathbb{Z}_2^2 & \underset{\cong}{\xrightarrow{(\Sigma^8 q_3)^*}} & [C_{\Sigma^8\omega},\mathrm{S}^7] & \longrightarrow & 0 & &
\end{array}$$

Thus we easily have

$$[C_{\Sigma^8\omega},\mathrm{Sp}(2)]=\mathbb{Z}_2\big\{(\Sigma^8 q_3)^*[\sigma'\eta_{14}]\big\}\oplus\mathbb{Z}_2\big\{i_*\overline{\varepsilon_3}\big\}.$$

5. $\pi_1\mathrm{map}_*(G_2,G_2)$

In this section we shall compute $[\Sigma G_2,G_2]$ ($\cong\pi_1\mathrm{map}_*(G_2,G_2)$). As in the subsection 3.7, we use the fibration

$$\mathrm{SU}(3)\xrightarrow{\hat{i}} G_2\xrightarrow{\hat{p}}\mathrm{S}^6,$$

and the following results from [**M**].

n	$\pi_n G_2$	gen. of 2-comp.
1,2,4,5,7,10,12,13	0	
3	$\mathbb{Z}$	$\hat{i}_*\iota_3$
6	$\mathbb{Z}_3$	
8	$\mathbb{Z}_2$	$\langle\eta_6^2\rangle$
9	$\mathbb{Z}_6$	$\langle\eta_6^2\rangle\circ\eta_8$
11	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\langle 2\Delta\iota_{13}\rangle, \hat{i}_*[\nu_5^2]$
14	$\mathbb{Z}_{168}\oplus\mathbb{Z}_2$	$\langle\bar{\nu}_6+\epsilon_6\rangle, \hat{i}_*[\nu_5^2]\circ\nu_{11}$
15	$\mathbb{Z}_2$	$\langle\bar{\nu}_6+\epsilon_6\rangle\circ\eta_{14}$

Table 5 : $\pi_n(G_2)$

In the Table 5 we follow the notations in [**M**].

As is well-known, G_2 has the cell structure:

$$G_2 = \mathrm{S}^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

Let $G_2^{(n)}$ denote the n-skeleton of G_2. Let $M^n = C_{2\iota_{n-1}} = \mathrm{S}^{n-1}\cup_{2\iota_{n-1}} e^n$ for $n\geq 2$, and

$$\mathrm{S}^{n-1} \xrightarrow{i_n} M^n \xrightarrow{q_n} \mathrm{S}^n$$

be the inclusion and the quotient map, respectively. Remark that $\Sigma M^n = M^{n+1}$. Then there exist the cofibrations as follows.

$$\mathrm{S}^3 \to G_2^{(6)} \xrightarrow{\pi_1} M^6, \tag{5.1}$$

$$G_2^{(6)} \to G_2^{(9)} \xrightarrow{\pi_2} M^9 \xrightarrow{\delta} \Sigma G_2^{(6)}. \tag{5.2}$$

From (5.1) we obtain [**MS**, Lemma 3.6]:

Lemma 5.1 ([**MS**])**.** $[\Sigma G_2^{(6)}, G_2] = 0$.

Next we shall show the following.

Lemma 5.2. $\Sigma\pi_2{}^* : [M^{10}, G_2] \to [\Sigma G_2^{(9)}, G_2]$ *is an isomorphism.*

Proof. From Lemma 5.1 it suffices to show that $(\Sigma\delta)^* : [\Sigma^2 G_2^{(6)}, G_2] \to [\Sigma M^9, G_2]$ is trivial. By Table 5 we easily have

$$[\Sigma M^9, G_2] = \mathbb{Z}_2\{\langle\eta_6^2\rangle\circ\overline{\eta_8}\}, \quad (\Sigma i_9)^*(\langle\eta_6^2\rangle\circ\overline{\eta_8}) = \langle\eta_6^2\rangle\circ\eta_8 \tag{5.3}$$

and

$$\pi_8(G_2) \xrightarrow[\cong]{(\Sigma^2 q_6)^*} [\Sigma^2 M_6, G_2] \xrightarrow[\cong]{(\Sigma^2\pi_1)^*} [\Sigma^2 G_2^{(6)}, G_2].$$

Hence it suffices to to prove the following equality:

$$(\Sigma i_9)^*(\Sigma\delta)^*(\Sigma^2\pi_1)^*(\Sigma^2 q_6)^*\langle\eta_6^2\rangle = 0.$$

We shall prove this by showing

$$\Sigma^2 q_6 \circ \Sigma^2\pi_1 \circ \Sigma\delta \circ \Sigma i_9 = 0 \in \pi_9(\mathrm{S}^8) = \mathbb{Z}_2\{\eta_8\}. \tag{5.4}$$

By [**Mu**], we have the following results.

$$[M^{10}, S^8] = \mathbb{Z}_4\{\overline{\eta_8}\}, \quad 2\overline{\eta_8} = \eta_8^2 \circ q_{10}, \tag{5.5}$$

$$[M^{10}, M^8] \cong \mathbb{Z}_2^3. \tag{5.6}$$

We have $2(\Sigma^2\pi_1 \circ \Sigma\delta) = 0$ by (5.6). Hence it follows from (5.5) that $\Sigma^2 q_6 \circ \Sigma^2\pi_1 \circ \Sigma\delta$ is divisible by 2. Thus (5.4) is established. □

Next we shall show that

Lemma 5.3. (1) *The induced map*

$$\Sigma i_{9,11}^* : [\Sigma G_2^{(11)}, G_2] \to [\Sigma G_2^{(9)}, G_2]$$

is an isomorphism, where $i_{9,11} : G_2^{(9)} \to G_2^{(11)}$ is the inclusion.

(2) $[\Sigma G_2^{(11)}, G_2] = \mathbb{Z}_2\left\{\overline{\overline{\langle\eta_6^2\rangle \circ \overline{\eta_8} \circ \Sigma\pi_2}}\right\}$.

Proof. The assertion (1) follows from $\pi_{12}(G_2) = 0$ ([**M**]) and [**MS**, Lemmas 3.9 (i) and 3.11] using the cofibration

$$\mathrm{S}^{10} \longrightarrow G_2^{(9)} \xrightarrow{i_{9,11}} G_2^{(11)}.$$

The assertion (2) follows from (1), (5.3) and Lemma 5.2. □

Let $f : \mathrm{S}^{13} \to G_2^{(11)}$ denote the attaching map of the top cell of G_2.

Lemma 5.4. *There exists the following short exact sequence.*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow [\Sigma G_2, G_2] \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \tag{5.7}$$

Proof. In the exact sequence induced by the cofibration $\mathrm{S}^{13} \xrightarrow{f} G_2^{(11)} \subset G_2$

$$[\Sigma^2 G^{(11)}, G_2] \xrightarrow{(\Sigma^2 f)^*} \pi_{15}(G_2) \xrightarrow{(\Sigma q)^*} [\Sigma G_2, G_2] \longrightarrow [\Sigma G_2^{(11)}, G_2] \xrightarrow{(\Sigma f)^*} \pi_{14}(G_2) \tag{5.8}$$

$(\Sigma f)^*$ is trivial by [**MS**, Lemma 3.13]. Here $q : G_2 \to \mathrm{S}^{14}$ is the quotient map. We show that $(\Sigma^2 f)^*$ is also trivial. To prove this, first we recall that

$$\pi_{15}(G_2) = \mathbb{Z}_2\{\langle\bar\nu_6 + \varepsilon_6\rangle \circ \eta_{14}\}$$

from [**M**]. Here $\langle\bar\nu_6 + \varepsilon_6\rangle$ is an element of $\pi_{14}(G_2)$ such that $\hat p_*\langle\bar\nu_6 + \varepsilon_6\rangle = \bar\nu_6 + \varepsilon_6$ by the bundle projection map $\hat p : G_2 \to \mathrm{S}^6$. By [**T2**, Lemma 6.3, Theorem 7.2], $(\bar\nu_6 + \varepsilon_6) \circ \eta_{14}$ is

stably nontrivial and so is $\langle\bar{\nu}_6+\varepsilon_6\rangle\circ\eta_{14}$. On the other hand, the attaching map f is stably trivial by [**BS**]. This means

$$\mathrm{Im}\ (\Sigma^2 f)^* = 0$$

in (5.8). Thus by (5.8), Lemma 5.2 and Lemma 5.3, we obtain the result. □

Theorem 5.5.

$$[\Sigma G_2, G_2] = \mathbb{Z}_2\{\langle\bar{\nu}_6+\varepsilon_6\rangle\circ\eta_{14}\circ\Sigma q\}\oplus\mathbb{Z}_2\Big\{\overline{\overline{\langle\eta_6^2\rangle\circ\overline{\eta_8}\circ\Sigma\pi_2}}\Big\}.$$

Proof. By Lemma 5.4, $[\Sigma G_2, G_2]$ is isomorphic to $\mathbb{Z}_2^2$ or $\mathbb{Z}_4$. To induce a contradiction, assume that it is isomorphic to $\mathbb{Z}_4$. In this case, by Lemma 5.3 (2) and the proof of Lemma 5.4, we have

$$2\,\overline{\overline{\langle\eta_6^2\rangle\circ\overline{\eta_8}\circ\Sigma\pi_2}} = \langle\bar{\nu}_6+\epsilon_6\rangle\circ\eta_{14}\circ\Sigma q.$$

Let $\ell : \{\Sigma G_2, G_2\}\to\pi_{15}^s(G_2)$ be a left inverse for $\Sigma^\infty q^* : \pi_{15}^s(G_2)\to\{\Sigma G_2, G_2\}$. It exists, because $\Sigma^\infty f = 0$. Here $\{X,Y\} = \lim_{n\to\infty}[\Sigma^n X, \Sigma^n Y]$ and $\pi_n^s(X) = \{\mathrm{S}^n, X\}$. We then have

$$\begin{aligned}
2\,\Sigma^\infty\hat{p}_*\circ\ell\Big(\Sigma^\infty\overline{\overline{\langle\eta_6^2\rangle\circ\overline{\eta_8}\circ\Sigma\pi_2}}\Big) &= \Sigma^\infty\hat{p}_*\circ\ell\Big(2\Sigma^\infty\overline{\overline{\langle\eta_6^2\rangle\circ\overline{\eta_8}\circ\Sigma\pi_2}}\Big)\\
&= \Sigma^\infty\hat{p}_*\circ\ell\circ\Sigma^\infty q^*(\langle\bar{\nu}+\epsilon\rangle\circ\eta)\\
&= (\bar{\nu}+\epsilon)\eta\\
&= \eta^2\sigma.
\end{aligned}$$

Note that the element $2\Sigma^\infty\hat{p}_*\circ\ell\Big(\Sigma^\infty\overline{\overline{\langle\eta_6^2\rangle\circ\overline{\eta_8}\circ\Sigma\pi_2}}\Big)$ is trivial since $\pi_9^s(S^0)\cong\mathbb{Z}_2^3$ ([**T2**]). This contradicts $\eta^2\sigma\neq 0$ ([**T2**]). Therefore, the short exact sequence (5.7) splits and we obtain the result. □

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